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Thermal partition functions for S-branes

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Abstract: We calculate the thermal partition functions of open strings on the S-brane backgrounds (the bouncing or rolling tachyon backgrounds) both in the bosonic and superstring cases. According to [9], we consider the discretized temperatures compatible with the pure imaginary periodicity of tachyon profiles. The "effective Hagedorn divergence" is shown to appear no matter how low temperature is chosen (including zero-temperature). This feature is likely to be consistent with the large rate of open string pair production discussed in [5] and also emission of closed string massive modes [8]. We also discuss the possibility to remove the divergence by considering the space-like linear dilaton backgrounds as in [27].

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, D-branes, Tachyon Condensation.

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1. Introduction

Time dependent physics in string theory has been getting more and more attentions and discussed by many theoretists from various angles. The S(pace-like) brane [1] has been providing an important laboratory to examine the real time dynamics of brane decay/creation processes caused by the open string tachyon condensation. Espicially, the time-like boundary Sinh-Gordon type interaction [2, 3, 4] $T(X^0) = \lambda \int_{\partial\Sigma} d\tau \, \cosh(X^0/\sqrt{\alpha'})$ and the Liouville type interaction [5, 6] $T(X^0) = \lambda \int_{\partial\Sigma} d\tau \, e^{\pm X^0/\sqrt{\alpha'}}$ have been intensively studied as the exactly soluble models of boundary conformal field theory (BCFT) describing the decay/creation processes of unstable D-branes. Following the terminology used in [1, 5], we shall call the Sinh-Gordon type model as the "full S-brane" and the Liouville type model as the "half S-brane".

Among other things, it has been discovered that these time dependent systems are followed by large rate of open string pair production [5, 7] and closed string emission [8]. It is an important fact that *all* the highly massive modes of string spectrum are excited by the

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decaying brane, which lead to the Hagedorn like UV divergence and could destabilize the gas of "tachyon matter" proposed in [2, 3, 4] based on the classical analysis. The existence of such Hagedorn like divergence has been already suggested in [5] and elucidated in [9]. Related discussions about the particle creation and UV instability in the S-brane system are also found in the papers e.g. [10]–[19].

Another remarkable point in the S-brane physics is the fact that we can define the thermal model *in spite of rapid time dependence,* if assuming the discretized temperature compatible with the imaginary periodicity of the boundary interaction [9].

Motivated by these studies, in this paper we calculate the thermal partition functions for the S-brane systems under the free string limit $g_s \to 0$, that is, the euclidean cylinder amplitudes with the thermal compactifications compatible with the rolling (or bouncing) tachyon profiles. We will further examine the thermodynamical behaviors of the partition functions. Although our conformal system is manifestly positive definite because of the euclidean signature, the modulus integral will be turnd out to show the Hagedorn like divergence *no matter how low temperature is taken.* This aspect seems to be consistent at least qualitatively with the observations given in [5, 7, 8] based on the caluculations in the lorentzian signature. The recent works developing exact world-sheet analysis in the lorentzian signature are given in [7, 20, 21, 22, 23, 24, 25, 26, 27], in particular, with emphasizing interesting relations to the old (boundary) Liouville theory of two dimensional gravity and matrix models.

This paper is organized as follows. Section 2 is the preliminary section. We summarize some aspects of the $c = 1$ ($c = 3/2$) boundary (super)conformal theory studied in [28, 29, 30]. Although nothing new is included in this section, we clarify some useful formulas to compute the cylinder amplitudes. In section 3, starting with presenting the thermal boundary states for the S-brane from the view points of affine SU(2)-current algebra, we calculate the exact thermal partition function. In section 4, we investigate the thermal behaviors of the partition functions, and find the strong thermal instabilities irrespective of the temperature we choose. We also discuss the possibility to remove the UV divergence in our euclidean amplitudes by considering the space-like linear dilaton backgrounds as in the recent paper [27]. Section 5 is devoted to presenting some discussions.

2. Preliminary: some notes on boundary conformal field theory

In this preliminary section we present a summary on the boundary (super) conformal theory with $c = 1$ ($c = 3/2$) studied in [28, 29, 30] for our later convenience. Although nothing new is included in this section, we will clarify some useful relations between the Virasoro Ishibashi states and those for the affine SU(2)-current algebra. The former is convenient to explicitly read off the interactions of S-branes with the closed string modes and has been treated in many literature [2, 3, 4, 31, 32, 12, 6, 33]. On the other hand, the latter is useful for the calculation of cylinder amplitudes (especially, in the thermal model we are interested in), as will be turned out in the next section.

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2.1 $c = 1$ boundary conformal field theory

We consider the free boson X compactified on the circle of self-dual radius; $X \sim X + 2\pi$.[1] As is well-known, we can realize the chiral $\mathrm{SU}(2)_1$ current algebra in this set-up. Decomposing to the chiral sectors as $X(z,\bar{z}) = X_L(z) + X_R(\bar{z})$, we define the $\mathrm{SU}(2)_1$ currents as

$$\begin{aligned} J^3(z) &= i\partial X_L(z)\,, \qquad J^{\pm}(z)\left(\equiv J^1(z) \pm iJ^2(z)\right) = e^{\pm i2X_L(z)}\,, \\ \tilde{J}^3(\bar{z}) &= i\bar{\partial} X_R(\bar{z})\,, \qquad \tilde{J}^{\pm}(\bar{z})\left(\equiv \tilde{J}^1(\bar{z}) \pm i\tilde{J}^2(\bar{z})\right) = e^{\pm i2X_R(\bar{z})}\,. \end{aligned} \tag{2.1}$$

The Neumann and Dirichlet boundary states are defined in the standard manner

$$|N\rangle = \frac{1}{2^{1/4}} \sum_{w\in\mathbf{Z}} \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right\} |w\rangle_L \otimes |-w\rangle_R\,, \tag{2.2}$$

$$|D\rangle = \frac{1}{2^{1/4}} \sum_{p\in\mathbf{Z}} \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right\} |p\rangle_L \otimes |p\rangle_R\,, \tag{2.3}$$

where α_n, $\tilde{\alpha}_n$ are the mode oscillators of $X(z)$, $\tilde{X}(\bar{z})$. The normalization factor $1/2^{1/4}$ is correctly chosen by the standard Cardy condition [34].

It is quite useful for our later analysis to introduce the Ishibashi states for the $\mathrm{SU}(2)_1$ current algebra [35]. They are defined associated to each of the integrable representations ($\ell = 0,1$), and characterized by the conditions

$$(J^a_n + \tilde{J}^a_{-n})\,|\ell\rangle\rangle = 0\,, \qquad ({}^{\forall}a, n) \tag{2.4}$$

$$\begin{aligned} \langle\langle \ell|\, e^{-\pi s H^{(c)}} e^{2\pi i z J^3_0}\, |\ell'\rangle\rangle &= \delta_{\ell,\ell'}\chi^{(1)}_{\ell}(is,z) \\ &\equiv \delta_{\ell,\ell'}\frac{\Theta_{\ell,1}(is,z)}{\eta(is)}\,, \end{aligned} \tag{2.5}$$

where $H^{(c)} \equiv L_0 + \tilde{L}_0 - 1/12$ is the closed string hamiltonian, and $\chi^{(1)}_{\ell}(\tau,z)$ denotes the $\mathrm{SU}(2)_1$ character of spin $\ell/2$ ($\ell = 0,1$). It is also often convenient to consider the Ishibashi states for the Virasoro algebra associated to the primary states of the degenerate representations: $|j,m,m'\rangle$ ($j \in \frac{1}{2}\mathbf{Z}_{\geq 0}$), which compose the spin j representation of the $\mathrm{SU}(2)$ zero-mode algebras J^a_0, $\tilde{J}^a_0$;

$$\begin{aligned} L_0\big|j,m,m'\big\rangle &= \tilde{L}_0\big|j,m,m'\big\rangle = j^2\big|j,m,m'\big\rangle\,, \\ L_n\big|j,m,m'\big\rangle &= \tilde{L}_n\big|j,m,m'\big\rangle = 0\,, \quad ({}^{\forall}n > 0)\,, \\ J^3_0\big|j,m,m'\big\rangle &= m\big|j,m,m'\big\rangle\,, \qquad \tilde{J}^3_0\big|j,m,m'\big\rangle = m'\big|j,m,m'\big\rangle\,. \end{aligned} \tag{2.6}$$

[1] We use the convention $\alpha' = 1$ throughout this paper. Namely, we have the OPE;

$$X_L(z)X_L(w) \sim -\frac{1}{2}\ln(z-w)$$

.

The corresponding Ishibashi states are defined by

$$
\begin{aligned}
(L_n - \tilde{L}_{-n})\left|j,m,m'\right\rangle\!\rangle &= 0\,, \qquad ({}^\forall n)\,,\\
\langle\!\langle j_1,m_1,m_1'|e^{-\pi s H^{(c)}}e^{2\pi i(zJ_0^3+\bar{z}\tilde{J}_0^3)}\left|j_2,m_2,m_2'\right\rangle\!\rangle &= \delta_{j_1,j_2}\delta_{m_1,m_2}\delta_{m_1',m_2'}\,\chi_j^{\mathrm{Vir}}(is)e^{2\pi i z m_1}e^{2\pi i\bar{z}m_1'}\\
&\equiv \delta_{j_1,j_2}\delta_{m_1,m_2}\delta_{m_1',m_2'}\,\frac{\tilde{q}^{j^2}-\tilde{q}^{(j+1)^2}}{\eta(is)}\times\\
&\quad\times e^{2\pi i z m_1}e^{2\pi i\bar{z}m_1'}\,, \qquad (2.7)
\end{aligned}
$$

where we wrote $\tilde{q}\equiv e^{-2\pi s}$ and $\chi_j^{\mathrm{Vir}}(is)$ denotes the Virasoro character of the degenerate representation with $h=j^2$. The decomposition of the affine character $\chi_\ell^{(1)}(\tau,z)$ by $\chi_j^{\mathrm{Vir}}(\tau)$ implies the obvious relations (under suitable choices of the phase factors of Ishibashi states);

$$
|0\rangle\!\rangle = \sum_{j\in\mathbf{Z}_{\geq 0}}\sum_{m=-j}^{j}|j,m,-m\rangle\!\rangle\,, \qquad |1\rangle\!\rangle = \sum_{j\in\frac{1}{2}+\mathbf{Z}_{\geq 0}}\sum_{m=-j}^{j}|j,m,-m\rangle\!\rangle\,. \qquad (2.8)
$$

We can also easily show that

$$
|N\rangle = \frac{1}{2^{1/4}}\left(|0\rangle\!\rangle + |1\rangle\!\rangle\right)\,, \qquad (2.9)
$$

$$
e^{2\pi i J_0^3}|N\rangle = \frac{1}{2^{1/4}}\left(|0\rangle\!\rangle - |1\rangle\!\rangle\right)\,, \qquad (2.10)
$$

and $|N\rangle$, $e^{2\pi i J_0^3}|N\rangle$ compose the complete basis of $\mathrm{SU}(2)_1$ Cardy states [34] defined by the boundary condition (2.4).

The Dirichlet boundary state (2.3) is similarly described by introducing the "twisted" Ishibashi states

$$
\widehat{|\ell\rangle\!\rangle} \stackrel{\mathrm{def}}{=} e^{-i\pi J_0^1}|\ell\rangle\!\rangle \equiv \sum_{j\in\frac{\ell}{2}+\mathbf{Z}_{\geq 0}} e^{-i\pi j}\sum_{m=-j}^{j}|j,m,m\rangle\!\rangle\,, \qquad (2.11)
$$

which satisfies the boundary condition

$$
(J_n^3 - \tilde{J}_{-n}^3)\widehat{|\ell\rangle\!\rangle} = 0\,, \qquad (J_n^\pm + \tilde{J}_{-n}^\mp)\widehat{|\ell\rangle\!\rangle} = 0\,, \qquad ({}^\forall n)\,. \qquad (2.12)
$$

We can then show

$$
|D\rangle = e^{-i\pi J_0^1}|N\rangle = \frac{1}{2^{1/4}}\left(\widehat{|0\rangle\!\rangle} + \widehat{|1\rangle\!\rangle}\right)\,, \qquad (2.13)
$$

by a little calculation. With the definition (2.11) we also obtain

$$
e^{i\pi J_0^1}|0\rangle\!\rangle = \widehat{|0\rangle\!\rangle}\,, \qquad e^{i\pi J_0^1}|1\rangle\!\rangle = -\widehat{|1\rangle\!\rangle}\,, \qquad (2.14)
$$

and hence

$$
\left|D'\right\rangle \equiv e^{i\pi J_0^1}|N\rangle = \frac{1}{2^{1/4}}\left(\widehat{|0\rangle\!\rangle} - \widehat{|1\rangle\!\rangle}\right)\,. \qquad (2.15)
$$

(2.13) and (2.15) again compose the basis of $\mathrm{SU}(2)_1$ Cardy states associated to the twisted boundary condition (2.12). The former corresponds to the periodic array of D-branes located at $X=2\pi n$ ($n\in\mathbf{Z}$), while the latter does to the configuration $X=2\pi(n+1/2)$ ($n\in\mathbf{Z}$). In fact, the twisted boundary condition (2.12) is equivalent with $J^3(z)=-\tilde{J}^3(\bar{z})$, $J^\pm(z)=\tilde{J}^\mp(\bar{z})$ in the open string channel, implying that $X\equiv X_L+X_R=2\pi n$ or $X\equiv X_L+X_R=2\pi(n+1/2)$ ($n\in\mathbf{Z}$).

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2.2 $c = 3/2$ boundary superconformal field theory

In order to generalize to the superstring case let us consider the $\mathcal{N}=1$ boundary superconformal theory of $c = 3/2$, described by a system of one free boson and fermion X, ψ. We assume the boson X is compactified on a circle with the free fermion radius; $X \sim X + 2\sqrt{2}\pi$. The superconformal system is now described by three free fermions, or equivalently, the $\mathrm{SU}(2)_2 (\cong \mathrm{SO}(3)_1)$ current algebra;

$$\begin{aligned} J^3(z) &= i\sqrt{2}\partial X_L(z)\,, \qquad & J^{\pm}(z) &= \pm i\sqrt{2}\psi_L e^{\pm i\sqrt{2}X_L}(z)\,, \\ \tilde{J}^3(z) &= i\sqrt{2}\bar{\partial} X_R(\bar{z})\,, \qquad & \tilde{J}^{\pm}(z) &= \pm i\sqrt{2}\psi_R e^{\pm i\sqrt{2}X_R}(\bar{z})\,. \end{aligned} \tag{2.16}$$

(The overall phase factors $\pm i$ in $J^{\pm}$ are chosen nothing but for the later convenience.)

The Neumann and Dirichlet boundary states are written as

$$|N\rangle = \exp\left\{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n} - i\sum_{r>0}\psi_{-r}\tilde{\psi}_{-r}\right\}|N\rangle^{(0)}\,, \tag{2.17}$$

$$|D\rangle = \exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n} + i\sum_{r>0}\psi_{-r}\tilde{\psi}_{-r}\right\}|D\rangle^{(0)}\,, \tag{2.18}$$

where r runs over $1/2 + \mathbf{Z}$ ($\mathbf{Z}$) for NS (R) sector, and $|N\rangle^{(0)}$, $|D\rangle^{(0)}$ denote the zero-mode parts specified later.[2]

As in the case of bosonic string it is useful to reexpress things by the language of SU(2) current algebra. We introduce the Ishibashi states $|\ell\rangle\rangle$ ($\ell = 0, 1, 2$) for $\mathrm{SU}(2)_2$ as the ones satisfy the boundary conditions (2.4) and also

$$\langle\langle \ell |\, e^{-\pi s H^{(c)}} e^{2\pi i z J_0^3}\, |\ell'\rangle\rangle = (-1)^{\ell}\delta_{\ell,\ell'}\chi_{\ell}^{(2)}(is, z)\,, \tag{2.19}$$

$$\begin{aligned} \chi_0^{(2)}(\tau, z) &\equiv \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau,0)}{\eta(\tau)}}\frac{\theta_3(\tau,z)}{\eta(\tau)} + \sqrt{\frac{\theta_4(\tau,0)}{\eta(\tau)}}\frac{\theta_4(\tau,z)}{\eta(\tau)}\right), \\ \chi_2^{(2)}(\tau, z) &\equiv \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau,0)}{\eta(\tau)}}\frac{\theta_3(\tau,z)}{\eta(\tau)} - \sqrt{\frac{\theta_4(\tau,0)}{\eta(\tau)}}\frac{\theta_4(\tau,z)}{\eta(\tau)}\right), \\ \chi_1^{(2)}(\tau, z) &\equiv \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau,0)}{\eta(\tau)}}\frac{\theta_2(\tau,z)}{\eta(\tau)}\,, \end{aligned} \tag{2.20}$$

where $\chi_{\ell}^{(2)}(\tau, z)$ denotes the spin $\ell/2$ character of $\mathrm{SU}(2)_2$. The overall factor $(-1)^{\ell}$ is the phase convention that reproduces the correct (-1) factor for the space-time fermions.

We may also consider the Ishibashi states for the $\mathcal{N}=1$ super Virasoro algebra $\{L_n, G_r\}$ defined similarly to (2.7). The degenerate representations are again characterized by the spin j of the SU(2) zero-mode algebra and have conformal weights $h = j^2/2$. To be

[2] The convention of boundary states with the boundary conditions $(\psi_r + i\epsilon\tilde{\psi}_{-r})|N;\epsilon\rangle = 0$, $(\psi_r - i\epsilon\tilde{\psi}_{-r})|D;\epsilon\rangle = 0$, $(\epsilon = \pm)$ is often used in literature, and the GSO projected boundary states are written as $\frac{1}{2}(|B;+\rangle - |B;-\rangle)$ ($B = N$ or D). In this paper we shall fix $\epsilon = +$ and explicitly insert the GSO projection operator when calculating amplitudes.

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more precise, we should restrict the spin j as $j \in \mathbf{Z}_{\geq 0}$ for the NS sector and $j \in \mathbf{Z}_{\geq 0}+1/2$ for the R sector due to the locality of the SU(2) currents (2.16). The corresponding Ishibashi states are defined by

$$
\begin{aligned}
(L_n - \tilde{L}_{-n})\,\big|j,m,m'\big\rangle\!\big\rangle &= 0\,, \quad ({}^{\forall} n)\,,\\
(G_r - i\tilde{G}_{-r})\,\big|j,m,m'\big\rangle\!\big\rangle &= 0\,, \quad \begin{cases} {}^{\forall} r \in \mathbf{Z}+\frac{1}{2} & \text{if } j \in \mathbf{Z}_{\geq 0} \\ {}^{\forall} r \in \mathbf{Z} & \text{if } j \in \mathbf{Z}_{\geq 0}+\frac{1}{2}\,. \end{cases}\\
\big\langle\!\big\langle j_1,m_1,m_1'\big|\, e^{-\pi s H^{(c)}} e^{2\pi i(zJ_0^3+\bar{z}\tilde{J}_0^3)}\,\big|j_2,m_2,m_2'\big\rangle\!\big\rangle &= \delta_{j_1,j_2}\delta_{m_1,m_2}\delta_{m_1',m_2'} \times\\
&\quad \times \chi_j^{\mathcal{N}=1}(is) e^{2\pi i z m_1} e^{2\pi i \bar{z} m_1'}\,, \qquad (2.21)\\
\chi_j^{\mathcal{N}=1}(\tau) &\equiv \frac{q^{\frac{j^2}{2}} - q^{\frac{(j+1)^2}{2}}}{\eta(\tau)}\sqrt{\frac{\theta_3}{\eta}(\tau)}\,, \quad (j \in \mathbf{Z}_{\geq 0})\\
\chi_j^{\mathcal{N}=1}(\tau) &\equiv \frac{q^{\frac{j^2}{2}} - q^{\frac{(j+1)^2}{2}}}{\eta(\tau)}\sqrt{\frac{2\theta_2}{\eta}(\tau)}\,, \quad \left(j \in \frac{1}{2}+\mathbf{Z}_{\geq 0}\right),
\end{aligned}
$$

where $\chi_j^{\mathcal{N}=1}(\tau)$ denotes the degenerate characters of $c=3/2$ super Virasoro algebra of NS (R) sector for $j \in \mathbf{Z}_{\geq 0}$ ($j \in 1/2+\mathbf{Z}_{\geq 0}$). Similarly to (2.8) the decomposition of the affine character $\chi_\ell^{(2)}(\tau,z)$ by $\chi_j^{\mathcal{N}=1}(\tau)$ leads to the relations

$$
\begin{aligned}
|0\rangle\!\rangle + |2\rangle\!\rangle &= \sum_{j \in \mathbf{Z}_{\geq 0}} \sum_{m=-j}^{j} |j,m,-m\rangle\!\rangle\,,\\
|1\rangle\!\rangle &= \sum_{j \in \frac{1}{2}+\mathbf{Z}_{\geq 0}} \sum_{m=-j}^{j} |j,m,-m\rangle\!\rangle\,. \qquad (2.22)
\end{aligned}
$$

Moreover, the Neumann boundary state of NS sector (2.17) is identified as

$$
|N\rangle_{NS} = |0\rangle\!\rangle + |2\rangle\!\rangle\,, \qquad (2.23)
$$

with the standard zero-mode part

$$
|N\rangle_{NS}^{(0)} = \sum_{w \in \mathbf{Z}} \left|\sqrt{2}w\right\rangle_L \otimes \left|-\sqrt{2}w\right\rangle_R. \qquad (2.24)
$$

We shall also define for the R sector

$$
|N\rangle_R = |1\rangle\!\rangle \qquad (2.25)
$$

for later convenience. It has the following zero-mode part (of bosonic sector)

$$
|N\rangle_R^{(0)} = \sum_{w \in \mathbf{Z}} \left|\frac{2w+1}{\sqrt{2}}\right\rangle_L \otimes \left|-\frac{2w+1}{\sqrt{2}}\right\rangle_R. \qquad (2.26)
$$

One might feel our definition of Ramond boundary state peculiar, since (2.26) does not have the standard winding modes compatible with the compactification $X \sim X + 2\sqrt{2}\pi$.

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Moreover, $|N\rangle_{NS}+|N\rangle_R$ is *not* the one describing the BPS D-brane. This choice of winding modes, however, make it possible for the $\mathrm{SU}(2)_2$ currents to act locally on the boundary state (2.25) (which is obvious since we identify $|N\rangle_R$ as the SU(2) Ishibashi state $|1\rangle\rangle$), and it will turn out later that this definition is actually useful to describe the S-brane boundary states in superstring.

The Dirichlet boundary states can be written in the similar manner to the bosonic case. We again have two possibilities

$$|D\rangle = e^{-i\pi J_0^1}\left(|N\rangle_{NS}+|N\rangle_R\right) \equiv e^{-i\pi J_0^1}\left(|0\rangle\rangle + |2\rangle\rangle + |1\rangle\rangle\right), \tag{2.27}$$

$$\begin{aligned} |D'\rangle &= e^{i\pi J_0^1}\left(|N\rangle_{NS}+|N\rangle_R\right) \equiv e^{i\pi J_0^1}\left(|0\rangle\rangle + |2\rangle\rangle + |1\rangle\rangle\right) \\ &\equiv e^{-i\pi J_0^1}\left(|0\rangle\rangle + |2\rangle\rangle - |1\rangle\rangle\right). \end{aligned} \tag{2.28}$$

The former (2.27) describes the alternating $D-\bar{D}$ brane array (under the suitable choice of phase factor);

$$\begin{cases} D\text{-branes} & \text{located at } X = \dfrac{2\pi}{\sqrt{2}}\left(2n+\dfrac{1}{2}\right) \quad (n\in\mathbf{Z}) \\ \bar{D}\text{-branes} & \text{located at } X = \dfrac{2\pi}{\sqrt{2}}\left(2n-\dfrac{1}{2}\right) \quad (n\in\mathbf{Z}) \end{cases} \tag{2.29}$$

and the latter (2.28) corresponds to the configuration with the inverse brane charges. We also note that in the open string channel the twisted boundary condition (2.12) leads to

$$\psi_L(z) = -\psi_R(\bar{z}), \qquad e^{\pm i\sqrt{2}X_L(z)} = -e^{\mp i\sqrt{2}X_R(\bar{z})}, \tag{2.30}$$

implying the brane array; $X\equiv X_L+X_R = \frac{2\pi}{\sqrt{2}}\left(n+\frac{1}{2}\right)$ $(n\in\mathbf{Z})$. Moreover, based on a careful observation for the RR sector, we can find the $D-\bar{D}$ configuration (2.29) with the expected periodicity $2\sqrt{2}\pi$.

3. Thermal partition functions for the S-branes

In this section we shall present our main analysis of the thermal partition functions for the S-brane backgrounds. We only consider the $g_s=0$ limit, in which the desired partition functions are given as the cylinder amplitudes in the euclidean signature. We start with the analysis in bosonic string case.

3.1 Bosonic string case

We first consider the full S-brane background (the bouncing tachyon profile), defined by the world-sheet action [2];

$$S_L = -\frac{1}{4\pi}\int_\Sigma d^2\sigma\,(\partial_\mu X^0)^2 + \lambda\int_{\partial\Sigma} d\tau\,\cosh X^0\,, \tag{3.1}$$

Based on the periodicity and physical reason discussed in [2], we restrict the range of the coupling λ to $0\le\lambda\le 1/2$ following [8]. The Wick-rotation $X^0 \to iX$ leads to the $c=1$ boundary conformal system studied in [28, 29, 30] in detail;

$$S_E = \frac{1}{4\pi}\int_\Sigma d^2\sigma\,(\partial_\mu X)^2 + \lambda\int_{\partial\Sigma} d\tau\,\cos X\,. \tag{3.2}$$

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(This is a tautological statement since the solution (3.1) was presented by Sen by taking the inverse Wick rotation of (3.2) from the beginning.) As is clarified in [28], the boundary interaction term is captured by the insertion of zero-mode of the SU(2) chiral current;

$$\lambda \int_{\partial\Sigma} d\tau \, \cos(X_L + X_R) \sim \lambda \oint dz \, \cos(2X_L) = 2\pi i \lambda J_0^1 \,, \tag{3.3}$$

where we used the Neumann boundary condition. We thus obtain the boundary state for the euclidean model with the compactification $X \sim X + 2\pi$ [28]

$$|B_{\lambda,1}\rangle \equiv e^{2\pi i \lambda J_0^1} |N\rangle \,, \tag{3.4}$$

where the Neumann boundary state $|N\rangle$ is defined in (2.2).

We can also consider more general compactification $X \sim X + 2\pi k$ ($k \in \mathbf{Z}_{>0}$), which is compatible with the locality of the SU(2) currents (2.1). This system is interpreted as the thermal model with the *discretized* temperature $T = 1/(2\pi k)$ compatible with the S-brane background, and the relevant boundary state is given [9] as

$$\begin{aligned} |B_{\lambda,k}\rangle &\equiv P_k \, e^{2\pi i \lambda J_0^1} |N\rangle \,, \\ &= \frac{1}{2^{1/4}} P_k \, e^{2\pi i \lambda J_0^1} \left(|0\rangle\rangle + |1\rangle\rangle \right) , \end{aligned} \tag{3.5}$$

where P_k is the projection operator to the Fock space with the momenta compatible with the thermal compactification;

$$(p_L, p_R) = \left(\frac{p}{k} + kw, \, \frac{p}{k} - kw \right) , \quad (p, w \in \mathbf{Z}) \,. \tag{3.6}$$

Note that $k = 2$ case corresponds to the Hagedorn temperature $T_H = 1/4\pi$ [36]. It is obvious that $P_1 = \mathrm{Id}$ and the zero-temperature case $k = \infty$ is described by the projection to the no winding Fock space $p_L = p_R$. The corresponding boundary state is the well-known one [28, 29, 30]

$$\begin{aligned} |B_{\lambda,\infty}\rangle &= P_\infty \, e^{2\pi i \lambda J_0^1} \left(|0\rangle\rangle + |1\rangle\rangle \right) \\ &= \sum_{j \in \frac{1}{2}\mathbf{Z}_{\geq 0}} \sum_{m=-j}^{j} D^j_{m,-m} \left(e^{2\pi i \lambda \frac{\sigma_1}{2}} \right) |j, m, m\rangle\rangle \,, \end{aligned} \tag{3.7}$$

where $D^j_{m,m'}$ denotes the Wigner function (the representation matrix of SU(2)). The boundary state of finite temperature is similarly written as [9]

$$|B_{\lambda,k}\rangle = \sum_{j \in \frac{1}{2}\mathbf{Z}_{\geq 0}} \sum_{m,w} D^j_{m-wk,-m} \left(e^{2\pi i \lambda \frac{\sigma_1}{2}} \right) |j, m, m - wk\rangle\rangle \,, \tag{3.8}$$

where the sum of m, w should be taken in the range such that $|m - wk| \leq j$.

Now, let us consider our main problem. We shall calculate the thermal cylinder amplitude of open strings ended at a single sp-brane, assuming the flat space-time for simplicity. The main part of calculation is the evaluation of the overlap along the euclidean time;

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$\langle B_{\lambda,k}|e^{-\pi s H^{(c)}}|B_{\lambda,k}\rangle$ with respect to the boundary states (3.5) (or (3.8)). This is in principle calculable, since all the things appearing in (3.8) are well-known quantities. However, the expressions (3.8) are not so useful for the relevant problem, since the Wigner function $D^j_{m,m'}$ has a quite complicated form in general (see for instance [30]). We shall thus take an other route. We calculate the amplitude based on the Ishibashi states *as the* SU(2) *current algebra* rather than the Virasoro Ishibashi states.

To begin with we note the relation

$$
\begin{aligned}
P_k\, e^{2\pi i\lambda J_0^1}\,|\ell\rangle\rangle &= \frac{1}{k}\sum_{r\in\mathbf{Z}_k} e^{2\pi i\frac{r}{k}(J_0^3-\tilde{J}_0^3)}e^{2\pi i\lambda J_0^1}\,|\ell\rangle\rangle \\
&= \frac{1}{k}\sum_{r\in\mathbf{Z}_k} e^{2\pi i\frac{r}{k}J_0^3}e^{2\pi i\lambda J_0^1}e^{2\pi i\frac{r}{k}J_0^3}\,|\ell\rangle\rangle\,,
\end{aligned}
\tag{3.9}
$$

which is the key identity for our calculation. In the second line we used the boundary condition (2.4). (We should emphasize that P_k is *not* equal $\frac{1}{k}\sum_{r\in\mathbf{Z}_k} e^{2\pi i\frac{r}{k}(J_0^3-\tilde{J}_0^3)}$ as an operator.)

Therefore, the overlap can be written as

$$
\begin{aligned}
\langle B_{\lambda,k}|e^{-\pi s H^{(c)}}|B_{\lambda,k}\rangle &= \frac{1}{\sqrt{2}}\sum_{\ell=0,1}\langle\langle\ell|\,e^{-2\pi i\lambda J_0^1}P_k e^{-\pi s H^{(c)}}P_k e^{2\pi i\lambda J_0^1}\,|\ell\rangle\rangle \qquad (3.10)\\
&= \frac{1}{\sqrt{2}}\frac{1}{k}\sum_{r\in\mathbf{Z}_k}\sum_{\ell=0,1}\langle\langle\ell|\,e^{-\pi s H^{(c)}}e^{-2\pi i\lambda J_0^1}e^{2\pi i\frac{r}{k}J_0^3}e^{2\pi i\lambda J_0^1}e^{2\pi i\frac{r}{k}J_0^3}\,|\ell\rangle\rangle\,,
\end{aligned}
$$

where we used the property of the projection operator; $[H^{(c)},\,P_k]=0$, $P_k^2=P_k$.

To proceed further we note that

$$
\begin{aligned}
&e^{-2\pi i\lambda\frac{\sigma_1}{2}}e^{2\pi i z\frac{\sigma_3}{2}}e^{2\pi i\lambda\frac{\sigma_1}{2}}e^{2\pi i z\frac{\sigma_3}{2}} = \\
&\quad= \begin{pmatrix} e^{i\pi z}\left(\cos\left(\frac{\pi r}{k}\right)+i\sin\left(\frac{\pi r}{k}\right)\cos(2\pi\lambda)\right) & -e^{-i\pi z}\sin\left(\frac{\pi r}{k}\right)\sin\left(2\pi\lambda\right) \\ e^{-i\pi z}\sin\left(\frac{\pi r}{k}\right)\sin\left(2\pi\lambda\right) & e^{-i\pi z}\left(\cos\left(\frac{\pi r}{k}\right)-i\sin\left(\frac{\pi r}{k}\right)\cos(2\pi\lambda)\right)\end{pmatrix} \\
&\quad= U\begin{pmatrix} e^{2\pi i\frac{1}{2}\alpha(\lambda,r/k)} & 0 \\ 0 & e^{-2\pi i\frac{1}{2}\alpha(\lambda,r/k)}\end{pmatrix}U^{-1}\,,
\end{aligned}
\tag{3.11}
$$

where $\alpha(\lambda,z)$ is a real number defined by

$$
\alpha(\lambda,z) = \frac{2}{\pi}\arcsin\left(\sin(\pi z)\cos(\pi\lambda)\right)\,,
\tag{3.12}
$$

and U denotes an unitary matrix (of which explicit form is not necessary for our argument). We thus find

$$
e^{-2\pi i\lambda J_0^1}e^{2\pi i\frac{r}{k}J_0^3}e^{2\pi i\lambda J_0^1}e^{2\pi i\frac{r}{k}J_0^3} = \mathcal{U}\,e^{2\pi i\alpha(\lambda,r/k)J_0^3}\,\mathcal{U}^{-1}\,,
\tag{3.13}
$$

where $\mathcal{U}$ is an unitary operator of the form such as $\mathcal{U}=e^{i\sum_a\theta^a(J_0^a+\tilde{J}_0^a)}e^{i\sum_a\theta^{a'}(J_0^a+\tilde{J}_0^a)}\cdots$. We also note the identity; $\mathcal{U}^{-1}|\ell\rangle\rangle = |\ell\rangle\rangle$ which is obvious from the boundary condition (2.4).

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In this way we can evaluate as

$$\begin{aligned}\langle B_{\lambda,k}|e^{-\pi sH^{(c)}}|B_{\lambda,k}\rangle &= \frac{1}{\sqrt{2}}\frac{1}{k}\sum_{r\in\mathbf{Z}_k}\sum_{\ell=0,1}\langle\langle \ell|\,e^{-\pi sH^{(c)}}e^{2\pi i\alpha(\lambda,r/k)J_0^3}\,|\ell\rangle\rangle \\ &= \frac{1}{\sqrt{2}}\frac{1}{k}\sum_{r\in\mathbf{Z}_k}\frac{1}{\eta(is)}\left\{\Theta_{0,1}(is,\alpha(\lambda,r/k))+\Theta_{1,1}(is,\alpha(\lambda,r/k))\right\} \\ &= \frac{1}{k}\sum_{r\in\mathbf{Z}_k}\frac{1}{\eta(it)}e^{-2\pi t\frac{1}{4}\alpha(\lambda,r/k)^2}\Theta_{0,1}\left(it, i\alpha\left(\lambda,\frac{r}{k}\right)t\right) \\ &= \frac{1}{k}\sum_{r\in\mathbf{Z}_k}\sum_{n\in\mathbf{Z}}\frac{1}{\eta(it)}e^{-2\pi t\left(n+\frac{1}{2}\alpha(\lambda,r/k)\right)^2}\,, \end{aligned} \tag{3.14}$$

where we have introduced the open string modulus $t \equiv 1/s$ and made use of the familiar modular property of theta function.

Taking account also of the spatial and ghost sectors, we obtain the final result of thermal partition function;[3]

$$Z(\lambda,k) = \mathcal{N}\int_0^\infty \frac{dt}{t}\frac{V_p}{(8\pi^2 t)^{p/2}}\frac{1}{\eta(it)^{24}}\frac{1}{k}\sum_{r\in\mathbf{Z}_k}\sum_{n\in\mathbf{Z}}e^{-2\pi t\left(n+\frac{1}{2}\alpha(\lambda,r/k)\right)^2} \tag{3.15}$$

where V_p is the volume of spatial part of sp-brane and $\mathcal{N}$ is the normalization factor which is determined just below. Under the zero-temperature limit $k\to\infty$, the summation of r is replaced with an integral, and we obtain

$$Z(\lambda,\infty) = \mathcal{N}\int_0^\infty \frac{dt}{t}\frac{V_p}{(8\pi^2 t)^{p/2}}\frac{1}{\eta(it)^{24}}\int_0^1 dz\sum_{n\in\mathbf{Z}}e^{-2\pi t\left(n+\frac{1}{2}\alpha(\lambda,z)\right)^2}\,, \tag{3.16}$$

which coincides the result already given in [28] (in the "note added in proof" in the NPB version as was pointed out in [25]). See also [29].

As a consistency check, let us focus on the special points $\lambda = 0$ and $\lambda = 1/2$. For $\lambda = 0$ we have $\alpha(0,r/k) = 2r/k$, leading to

$$\begin{aligned}\langle B_{0,k}|e^{-\pi sH^{(c)}}|B_{0,k}\rangle &= \frac{1}{k}\sum_{r\in\mathbf{Z}_k}\frac{1}{\eta(it)}\sum_{n\in\mathbf{Z}}e^{-2\pi t\left(n+\frac{r}{k}\right)^2} \\ &= \frac{1}{k}\frac{1}{\eta(it)}\sum_{n\in\mathbf{Z}}e^{-2\pi t\frac{n^2}{k^2}} \\ &= \frac{1}{k}\frac{1}{\eta(it)}\frac{k}{\sqrt{2t}}\sum_{n\in\mathbf{Z}}e^{-\frac{(2\pi kn)^2}{8\pi t}}\,, \end{aligned} \tag{3.17}$$

where we used the Poisson resummation formula in the last line. We thus obtain

$$Z(0,k) = \frac{\mathcal{N}}{k}\int_0^\infty \frac{dt}{t}\frac{2\pi kV_p}{(8\pi^2 t)^{\frac{p+1}{2}}}\frac{1}{\eta(it)^{24}}\sum_{n\in\mathbf{Z}}e^{-\frac{(2\pi kn)^2}{8\pi t}}\,. \tag{3.18}$$

[3] As we will clarify later, it turns out that the t-integral here includes UV and IR divergences for generic λ. Hence we should introduce the UV and IR cut-off's into the t-integral to be more rigorous.

As is expected, this is precisely the thermal partition function for a single (time-like) Dp-brane with temperature $T = 1/(2\pi k)$ up to normalization. Therefore, we should set $\mathcal{N} = k$ for the correct normalization.

For $\lambda = 1/2$ we have $\alpha(1/2, r/k) = 0$ and obtain

$$\langle B_{1/2,k} | e^{-\pi s H^{(c)}} | B_{1/2,k} \rangle = \frac{\Theta_{0,1}(it, 0)}{\eta(it)} . \tag{3.19}$$

The thermal partition function now becomes

$$Z\left(\frac{1}{2}, k\right) = k \int_0^\infty \frac{dt}{t} \frac{V_p}{(8\pi^2 t)^{p/2}} \frac{1}{\eta(it)^{24}} \sum_{n \in \mathbf{Z}} e^{-\frac{(2\pi n)^2 t}{2\pi}} . \tag{3.20}$$

This is of course consistent with the periodic array of $D(p-1)$ brane instantons at $X = 2\pi(n + 1/2)$ ($n \in \mathbf{Z}$) mentioned previously. Note that they have no support on the real time axis, as is expected since $\lambda = 1/2$ means that we are already sitting at the minimum of tachyon potential [2]. The overall factor k is consistent with the fact that we have k D-brane instanton on the thermal circle of radius k. Note that the amplitude in this case does not depend on the temperature $T = 1/(2\pi k)$ except for the normalization. This is due to the simple fact; $P_k \, \widehat{|\ell\rangle\rangle} = \widehat{|\ell\rangle\rangle}$ $({}^\forall k)$, and regarded as the first signal of the "effective thermalization" caused by S-brane discussed in [5, 7, 9].

The calculation for the half S-brane is almost parallel. We shall only focus on the brane decay solution $T(X^0) = \lambda e^{X^0}$. The world-sheet action is given by

$$S_L = -\frac{1}{4\pi} \int_\Sigma d^2\sigma \, (\partial_\mu X^0)^2 + \lambda \int_{\partial\Sigma} d\tau \, e^{X^0} . \tag{3.21}$$

The thermal boundary state is similarly constructed by inserting $e^{2\pi i \lambda J_0^+}$;

$$\left| B_{\lambda,k}^{(+)} \right\rangle = \frac{1}{2^{1/4}} P_k \, e^{2\pi i \lambda J_0^+} \left(|0\rangle\rangle + |1\rangle\rangle \right) . \tag{3.22}$$

The calculation of cylinder amplitude can be carried out in the same way. The most non-trivial point is

$$\begin{aligned} & e^{-2\pi i \lambda J_0^-} e^{2\pi i z J_0^3} e^{2\pi i \lambda J_0^+} e^{2\pi i z J_0^3} \sim e^{2\pi i \alpha^{(+)}(\lambda, r) J_0^3} , \\ & \alpha^{(+)}(\lambda, z) = \frac{1}{\pi} \arccos\left(\cos(2\pi z) + \frac{1}{2}(2\pi\lambda)^2 \right) , \end{aligned} \tag{3.23}$$

where $\sim$ means the equality up to a similarity transformation as in (3.13). Notice that $\alpha^{(+)}(\lambda, z)$ here is generally a complex number contrary to the full brane case. This is because $e^{2\pi i \lambda J_0^+}$ is not an unitary operator. To be more specific, we find

$$\begin{aligned} & \alpha^{(+)}(\lambda, z) \in \mathbf{R} \quad \text{if} \quad \left| \cos(2\pi z) + \frac{1}{2}(2\pi\lambda)^2 \right| \leq 1 , \\ & \alpha^{(+)}(\lambda, z) \in i\mathbf{R} \quad \text{if} \quad \left| \cos(2\pi z) + \frac{1}{2}(2\pi\lambda)^2 \right| > 1 . \end{aligned} \tag{3.24}$$

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The final result is written as

$$Z_{\rm half}(\lambda,k)=k\int_0^\infty\frac{dt}{t}\frac{V_p}{(8\pi^2t)^{p/2}}\frac{1}{\eta(it)^{24}}\frac{1}{k}\sum_{r\in\mathbf{Z}_k}\sum_{n\in\mathbf{Z}}e^{-2\pi t\left(n+\frac{1}{2}\alpha^{(+)}(\lambda,r/k)\right)^2}\,, \tag{3.25}$$

This is a real function as we can check it by using (3.24).

We here make an important comment. In the world-sheet action (3.21) we can absorb the coupling λ into the zero-mode of X^0. In other words, the coupling λ is *dynamical* in the lorentzian theory and should be integrated out when performing the path integration. In fact, the lorentzian cylinder amplitudes calculated in [8, 27] do not depend on λ as should be. On the other hand, the path integration is performed along the imaginary time axis in our thermal model. Rewriting as $\lambda\equiv\lambda_0 e^{x^0}$, we can regard the *real* time coordinate x^0 as a *parameter* of euclidean world-sheet action. The thermal partition function (3.25) now gains an explicit time dependence through the coupling λ.

3.2 Superstring case

Next let us turn to the superstring case. First we again consider the full S-brane, which is defined as the non-BPS D-brane with the tachyon profile $T(X^0)=\frac{\lambda}{\sqrt{2}}\cosh(X^0/\sqrt{2})$. The corresponding vertex operator in the zero-picture is given [3] as

$$\frac{\lambda}{\sqrt{2}}\int_{\partial\Sigma}d\tau\,\sigma_1\otimes\psi^0\sinh\left(\frac{X^0}{\sqrt{2}}\right)\,, \tag{3.26}$$

where σ_1 is the Chan-Paton (CP) factor characteristic for the non-BPS D-brane. The Wick-rotated world-sheet action is given by

$$\begin{aligned}S_E&=\frac{1}{4\pi}\int_\Sigma d^2\sigma\,(\partial_\mu X)^2-\frac{\lambda}{\sqrt{2}}\int_{\partial\Sigma}d\tau\,\sigma_1\otimes\psi\sin\left(\frac{X}{\sqrt{2}}\right)\,,\\&=\frac{1}{4\pi}\int_\Sigma d^2\sigma\,(\partial_\mu X)^2+i\frac{\lambda}{2\sqrt{2}}\int_{\partial\Sigma}d\tau\,\sigma_1\otimes\psi\left(e^{iX/\sqrt{2}}-e^{-iX/\sqrt{2}}\right)\,.\end{aligned} \tag{3.27}$$

Hence the boundary interaction is again described by the $\mathrm{SU}(2)_2$ current (2.16). However, the problem is slightly non-trivial compared with the bosonic case because of the existence of CP matrix σ_1. The boundary interaction should be realized as an insertion of Wilson line [37]. For the NSNS sector, the Wilson line is simply given as

$$\frac{1}{2}\mathrm{Tr}\left(e^{2\pi i\lambda\sigma_1\otimes J_0^1}\right)=\frac{1}{2}\left(e^{2\pi i\lambda J_0^1}+e^{-2\pi i\lambda J_0^1}\right)\,. \tag{3.28}$$

For the RR sector, on the other hand, we need to be more careful. We must insert the extra CP factor σ_1 into the trace so as to obtain the consistent result (see [38], for instance);

$$\frac{1}{2}\mathrm{Tr}\left(\sigma_1\,e^{2\pi i\lambda\sigma_1\otimes J_0^1}\right)=\frac{1}{2}\left(e^{2\pi i\lambda J_0^1}-e^{-2\pi i\lambda J_0^1}\right)\,. \tag{3.29}$$

As in the bosonic case, we can consider the thermal compactification $X\sim X+2\pi\sqrt{2}k$ ($k\in\mathbf{Z}_{>0}$) compatible with the boundary interaction. The zero-mode momenta should be restricted as

$$(p_L,p_R)=\left(\frac{p}{\sqrt{2}k}+\sqrt{2}kw,\,\frac{p}{\sqrt{2}k}-\sqrt{2}kw\right)\,,\qquad(p,w\in\mathbf{Z})\,, \tag{3.30}$$

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and we denote the associated thermal projection operator as P_k. Taking account of the Wilson line operators, the thermal boundary state is written as

$$
\begin{aligned}
|B_{\lambda,k}\rangle &= \frac{1}{2}P_k\left(e^{2\pi i\lambda J_0^1}+e^{-2\pi i\lambda J_0^1}\right)\sqrt{2}|N\rangle_{NS}+\frac{1}{2}P_k\left(e^{2\pi i\lambda J_0^1}-e^{-2\pi i\lambda J_0^1}\right)\sqrt{2}|N\rangle_{R} \\
&\equiv \frac{1}{\sqrt{2}}P_k\left(e^{2\pi i\lambda J_0^1}+e^{-2\pi i\lambda J_0^1}\right)(|0\rangle\rangle+|2\rangle\rangle)+\frac{1}{\sqrt{2}}P_k\left(e^{2\pi i\lambda J_0^1}-e^{-2\pi i\lambda J_0^1}\right)|1\rangle\rangle\,, \qquad (3.31)
\end{aligned}
$$

where the factor $\sqrt{2}$ is originating from the tension of the non-BPS D-brane. To confirm the validity of it we first point out that, under the zero temperature limit $k\to\infty$, the boundary state (3.31) correctly reproduces the NSNS and RR source terms of zero-mode sectors (in the lorentzian theory) presented in [3, 4];

$$
\begin{aligned}
f_{\rm NS}(x^0) &= \frac{1}{1+e^{\sqrt{2}x^0}\sin^2(\pi\lambda)}+\frac{1}{1+e^{-\sqrt{2}x^0}\sin^2(\pi\lambda)}-1\,, \\
f_{\rm R}(x^0) &= \sin(\pi\lambda)\left(\frac{e^{\frac{x^0}{\sqrt{2}}}}{1+e^{\sqrt{2}x^0}\sin^2(\pi\lambda)}-\frac{e^{-\frac{x^0}{\sqrt{2}}}}{1+e^{-\sqrt{2}x^0}\sin^2(\pi\lambda)}\right). \qquad (3.32)
\end{aligned}
$$

Secondly, let us focus on the special points $\lambda=0$ and $\lambda=1/2$ (or $\lambda=-1/2$). It is easily found that

$$
\begin{aligned}
|B_{0,k}\rangle &= \sqrt{2}P_k\,|N\rangle_{NS}\equiv\sqrt{2}P_k\,(|0\rangle\rangle+|2\rangle\rangle)\,, \\
\left|B_{1/2,k}\right\rangle &= \sqrt{2}e^{-i\pi J_0^1}(|N\rangle_{NS}+|N\rangle_{R})\equiv\sqrt{2}e^{-i\pi J_0^1}(|0\rangle\rangle+|2\rangle\rangle+|1\rangle\rangle)\equiv\sqrt{2}|D\rangle\,, \\
\left|B_{-1/2,k}\right\rangle &= \sqrt{2}e^{-i\pi J_0^1}(|N\rangle_{NS}-|N\rangle_{R})\equiv\sqrt{2}e^{-i\pi J_0^1}(|0\rangle\rangle+|2\rangle\rangle-|1\rangle\rangle)\equiv\sqrt{2}|D'\rangle\,. \qquad (3.33)
\end{aligned}
$$

(Recall the relations (2.27), (2.28).) The $\lambda=0$ case correctly reproduces the thermal boundary state of the non-BPS D-brane, and the $\lambda=\pm1/2$ cases correspond to the periodic $D-\bar{D}$ instanton configurations as is expected. In fact, if neglecting the projection P_k, the boundary state (3.31) is precisely the T-dual of the one describing the interpolation between the non-BPS $D0$-brane and $D1-\bar{D}1$ system (with the $\mathbf{Z}_2$ Wilson line) compactified on the critical radius $R=\sqrt{2}$ [39, 40, 38]. (This is again a tautological statement by construction of the rolling tachyon solution.) However, the existence of P_k is essential for our analysis and makes the cylinder amplitude quite different from that given in [40]. Note also that we have no k dependence when $\lambda=\pm1/2$. This is again originating from the fact that the thermal projection P_k trivially acts on the Dirichlet boundary states.

Now we are in the position to calculate the thermal partition function. To make use of the similar technique as in the bosonic case, we first note that

$$
\begin{aligned}
P_k\,e^{2\pi i\lambda J_0^1}\,|\ell\rangle\rangle &= \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}}e^{2\pi i\frac{r}{2k}(J_0^3-\tilde{J}_0^3)}e^{2\pi i\lambda J_0^1}\,|\ell\rangle\rangle \\
&= \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}}e^{2\pi i\frac{r}{2k}J_0^3}e^{2\pi i\lambda J_0^1}e^{2\pi i\frac{r}{2k}J_0^3}\,|\ell\rangle\rangle\,, \qquad (3.34)
\end{aligned}
$$

which is similar to (3.9) but includes slightly different coefficients. In addition to (3.13), (3.12), the following relation is useful;

$$
\left(e^{2\pi i\lambda J_0^1}e^{2\pi izJ_0^3}\right)^2\sim e^{2\pi i\beta(\lambda,z)J_0^3}\,, \qquad (3.35)
$$

where we set

$$\beta(\lambda, z) = \frac{2}{\pi} \arccos\left(\cos(\pi z)\cos(\pi\lambda)\right) . \tag{3.36}$$

Another non-trivial point is the thermal boundary condition for the fermionic sector given in [43]. In our present problem it amounts to the insertion of the "thermal GSO projection"

$$\frac{1}{2}\left(1 + (-1)^{F_L + W}\right) \equiv \frac{1}{2}\left(1 + (-1)^{F_L} e^{2\pi i \frac{1}{4k}(J_0^3 - \tilde{J}_0^3)}\right), \tag{3.37}$$

where F_L denotes the fermion number operator in the left mover and $W \equiv p_L - p_R$ stands for the winding along the thermal circle, instead of the usual GSO projection $\frac{1}{2}\left(1 + (-1)^{F_L}\right)$.[4] The following formulas are also useful;

$$\begin{aligned}
[(-1)^{F_L} e^{i\pi J_0^3},\, J_n^a] &= 0\,, \qquad (^\forall a, n)\,, \\
(-1)^{F_L} e^{i\pi J_0^3}\, |0\rangle\rangle &= -|0\rangle\rangle\,, \qquad (-1)^{F_L} e^{i\pi J_0^3}\, |2\rangle\rangle = |2\rangle\rangle\,, \\
(-1)^{F_L}\, |1\rangle\rangle &= 0\,.
\end{aligned} \tag{3.38}$$

(We take the standard convention $(-1)^{F_L}|0\rangle_{\mathrm{NS}} = -|0\rangle_{\mathrm{NS}}$ for the NS Fock vacuum.) The last identity is due to the fermion zero-modes in the Ramond sector.

In this way we can evaluate the overlap amplitude as follows;

$$\begin{aligned}
&{}_{\mathrm{NS}}\langle B_{\lambda,k}| e^{-\pi s H^{(c)}} |B_{\lambda,k}\rangle_{\mathrm{NS}} = \\
&= \frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \sum_{\ell=0,2} \Big\{ \langle\langle \ell |\, e^{-\pi s H^{(c)}} e^{2\pi i \alpha(\lambda, \frac{r}{2k}) J_0^3}\, |\ell\rangle\rangle + \langle\langle \ell |\, e^{-\pi s H^{(c)}} e^{2\pi i \beta(\lambda, \frac{r}{2k}) J_0^3}\, |\ell\rangle\rangle + \\
&\qquad\qquad + \langle\langle \ell |\, e^{-\pi s H^{(c)}} e^{2\pi i \alpha(-\lambda, \frac{r}{2k}) J_0^3}\, |\ell\rangle\rangle + \langle\langle \ell |\, e^{-\pi s H^{(c)}} e^{2\pi i \beta(-\lambda, \frac{r}{2k}) J_0^3}\, |\ell\rangle\rangle \Big\} \\
&= \frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \sum_{\ell=0,2} \left\{ \chi_\ell^{(2)}\left(is, \alpha\left(\lambda, \frac{r}{2k}\right)\right) + \chi_\ell^{(2)}\left(is, \beta\left(\lambda, \frac{r}{2k}\right)\right) \right\} \\
&= \frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \left\{ \sqrt{\frac{\theta_3(is,0)}{\eta(is)}}\, \frac{\theta_3(is, \alpha(\lambda, \frac{r}{2k}))}{\eta(is)} + \sqrt{\frac{\theta_3(is,0)}{\eta(is)}}\, \frac{\theta_3(is, \beta(\lambda, \frac{r}{2k}))}{\eta(is)} \right\} .
\end{aligned} \tag{3.39}$$

We likewise obtain for the RR sector

$$\begin{aligned}
&{}_{\mathrm{R}}\langle B_{\lambda,k}| e^{-\pi s H^{(c)}} |B_{\lambda,k}\rangle_{\mathrm{R}} = \\
&\quad = -\frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \left\{ \sqrt{\frac{\theta_2(is,0)}{2\eta(is)}}\, \frac{\theta_2(is, \alpha(\lambda, \frac{r}{2k}))}{\eta(is)} - \sqrt{\frac{\theta_2(is,0)}{2\eta(is)}}\, \frac{\theta_2(is, \beta(\lambda, \frac{r}{2k}))}{\eta(is)} \right\} .
\end{aligned} \tag{3.40}$$

The calculation of ${}_{\mathrm{NS}}\langle B_{\lambda,k}|(-1)^{F_L} e^{2\pi i \frac{1}{4k}(J_0^3 - \tilde{J}_0^3)} e^{-\pi s H^{(c)}} |B_{\lambda,k}\rangle_{\mathrm{NS}}$ is more non-trivial. With the helps of (3.38) we can make use of the identities such as

$$\begin{aligned}
\langle\langle 0 |\, e^{-2\pi i \lambda J_0^1} (-1)^{F_L} &= \langle\langle 0 |\, (-1)^{F_L} e^{2\pi i \lambda J_0^1} = -\langle\langle 0 |\, e^{i\pi J_0^3} e^{2\pi i \lambda J_0^1}\,, \\
\langle\langle 2 |\, e^{-2\pi i \lambda J_0^1} (-1)^{F_L} &= \langle\langle 2 |\, (-1)^{F_L} e^{2\pi i \lambda J_0^1} = \langle\langle 2 |\, e^{i\pi J_0^3} e^{2\pi i \lambda J_0^1}
\end{aligned} \tag{3.41}$$

[4] When calculating the overlap of boundary states, it is enough to consider the GSO projection only for the left mover (or the right mover) $\frac{1}{2}\left(1 + (-1)^{F_L}\right)$ in place of $\frac{1}{2}\left(1 + (-1)^{F_L}\right) \cdot \frac{1}{2}\left(1 + (-1)^{F_R}\right)$.

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and also introduce

$$e^{i\pi J_0^3} e^{-2\pi i\lambda J_0^1} e^{2\pi i z J_0^3} e^{2\pi i\lambda J_0^1} e^{2\pi i z J_0^3} \sim e^{i\pi J_0^3} (e^{2\pi i\lambda J_0^1} e^{2\pi i z J_0^3})^2 \sim e^{2\pi i\gamma(\lambda,z) J_0^3} ,$$
$$\gamma(\lambda, z) = \frac{1}{\pi} \arccos \left(\cos \left(\pi \left(2z + \frac{1}{2} \right) \right) \cos^2(\pi\lambda) \right) , \tag{3.42}$$

as in (3.13), (3.35). We thus obtain

$$\begin{aligned} &{}_{\rm NS}\langle B_{\lambda,k}| (-1)^{F_L} e^{2\pi i \frac{1}{4k}(J_0^3 - \tilde{J}_0^3)} e^{-\pi s H^{(c)}} |B_{\lambda,k}\rangle_{\rm NS} \\ &= \frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \left\{ -\chi_0^{(2)}(is, \gamma(\lambda, \frac{r+1/2}{2k})) + \chi_2^{(2)}(is, \gamma(\lambda, \frac{r+1/2}{2k})) \right\} \\ &= -\frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \sqrt{\frac{\theta_4(is,0)}{\eta(is)}} \frac{\theta_4(is, \gamma(\lambda, \frac{r+1/2}{2k}))}{\eta(is)} . \end{aligned} \tag{3.43}$$

Gathering contributions from all the sectors and performing the modular transformation to the open string modulus $t \equiv 1/s$, we finally obtain the following thermal partition function;

$$\begin{aligned} Z(\lambda, k) =& \mathcal{N} \int_0^\infty \frac{dt}{t} \frac{V_p}{(8\pi^2 t)^{p/2}} \frac{1}{\eta(it)^8} \frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \times \\ &\times \frac{1}{2} \Bigg[\left(\frac{\theta_3}{\eta}\right)^4 (it) \cdot \frac{1}{2} \Bigg\{ \theta_3 \left(it, i\alpha\left(\lambda, \frac{r}{2k}\right) t \right) e^{-2\pi t \frac{1}{2}\alpha(\lambda, \frac{r}{2k})^2} + \\ &\qquad + \theta_3 \left(it, i\beta\left(\lambda, \frac{r}{2k}\right) t \right) e^{-2\pi t \frac{1}{2}\beta(\lambda, \frac{r}{2k})^2} \Bigg\} - \\ &- \left(\frac{\theta_4}{\eta}\right)^4 (it) \cdot \frac{1}{2} \Bigg\{ \theta_4 \left(it, i\alpha\left(\lambda, \frac{r}{2k}\right) t \right) e^{-2\pi t \frac{1}{2}\alpha(\lambda, \frac{r}{2k})^2} - \\ &\qquad - \theta_4 \left(it, i\beta\left(\lambda, \frac{r}{2k}\right) t \right) e^{-2\pi t \frac{1}{2}\beta(\lambda, \frac{r}{2k})^2} \Bigg\} - \\ &- \left(\frac{\theta_2}{\eta}\right)^4 (it) \theta_2 \left(it, i\gamma\left(\lambda, \frac{r+1/2}{2k}\right) t \right) e^{-2\pi t \frac{1}{2}\gamma(\lambda, \frac{r+1/2}{2k})^2} \Bigg] , \end{aligned} \tag{3.44}$$

where $\mathcal{N}$ is a normalization constant which should be determined by the consistency with the $\lambda = 0$ case as in the bosonic case.

Let us consider the special cases $\lambda = 0$, $\lambda = 1/2$ which simplifies the amplitude. For $\lambda = 0$, since $\alpha(0, z) = 2z$, $\beta(0, z) = 2z$, $\gamma(0, z) = 2z + 1/2$ holds, we can easily find

$$\begin{aligned} Z(0, k) =& \frac{\mathcal{N}}{2} \int_0^\infty \frac{dt}{t} \frac{V_p}{(8\pi^2 t)^{p/2}} \frac{1}{\eta(it)^8} \frac{1}{2k} \sum_{r\in \mathbf{Z}_{2k}} \times \\ &\times \left[\left(\frac{\theta_3}{\eta}\right)^4 (it) \theta_3 \left(it, i\frac{r}{k} t \right) e^{-2\pi t \frac{1}{2}(\frac{r}{k})^2} - \left(\frac{\theta_2}{\eta}\right)^4 (it) \theta_3 \left(it, i\frac{r+1/2}{k} t \right) e^{-2\pi t \frac{1}{2}(\frac{r}{k})^2} \right] \\ =& \frac{\mathcal{N}}{2k} \int_0^\infty \frac{dt}{t} \frac{2\sqrt{2}\pi k V_p}{(8\pi^2 t)^{\frac{p+1}{2}}} \frac{1}{\eta(it)^8} \sum_{n\in \mathbf{Z}} e^{-\frac{(2\sqrt{2}\pi k n)^2}{8\pi t}} \times \\ &\times \left\{ \left(\frac{\theta_3}{\eta}\right)^4 (it) - (-1)^n \left(\frac{\theta_2}{\eta}\right)^4 (it) \right\} . \end{aligned} \tag{3.45}$$

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In the second line we have used the Poisson resummation formula. This is the correct form of thermal partition function for the non-BPS brane with the temperature $T = 1/(2\sqrt{2}\pi k)$ up to normalization. Especially, the correct phase factor $(-1)^n$ for the space-time fermions is successfully realized. In this way we should fix the normalization constant as $\mathcal{N} = 2k$.

For $\lambda = 1/2$, we have $\alpha(1/2, z) = 0$, $\beta(1/2, z) = 1$, and $\gamma(1/2, z) = 1/2$. We thus obtain after a little calculation

$$Z\left(\frac{1}{2}, k\right) = 2k \int_0^\infty \frac{dt}{t} \frac{V_p}{(8\pi^2 t)^{p/2}} \frac{1}{\eta(it)^8} \left(\frac{\theta_4}{\eta}\right)^4 (it) \sum_{n\in\mathbf{Z}} e^{-\frac{t}{2\pi}\{2\sqrt{2}\pi(n+1/2)\}^2} . \tag{3.46}$$

We here used the familiar Jacobi identity

$$\left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 - \left(\frac{\theta_2}{\eta}\right)^4 = 0 . \tag{3.47}$$

It is a straightforward task to confirm the amplitude (3.46) is really the one expected from the $D(p-1)-\bar{D}(p-1)$ system (2.29). The $D-D$ and $\bar{D}-\bar{D}$ open strings do not contribute to the amplitude by the SUSY cancellation, and the $D-\bar{D}$ strings yield the correct instanton action $\frac{t}{2\pi}\left\{2\sqrt{2}\pi(n+1/2)\right\}^2$. It may be amazing that the alternating structure of $D-\bar{D}$ system correctly reproduces the thermal boundary condition for fermions (that is implicit in the θ_4 factors in (3.46)). The overall factor $2k$ is the correct degeneracy of configuration on the thermal circle with radius $\sqrt{2}k$.

The thermal partition functions for the half S-branes are similarly computed with a little modification. We again only consider the brane decay case $T(X^0) = \frac{1}{2\sqrt{2}}\lambda e^{X^0/\sqrt{2}}$ (in the (-1) picture). In addition to the "twist angle" $\alpha^{(+)}(\lambda, z)$ defined in (3.23), we introduce $\beta^{(+)}(\lambda, z)$, $\gamma^{(+)}(\lambda, z)$ as follows;

$$\begin{aligned}
& e^{2\pi i\lambda J_0^-} e^{2\pi i z J_0^3} e^{2\pi i\lambda J_0^+} e^{2\pi i z J_0^3} \sim e^{2\pi i\beta^{(+)}(\lambda,z)J_0^3} , \\
& \beta^{(+)}(\lambda, z) = \frac{1}{\pi} \arccos\left(\cos(2\pi z) - \frac{1}{2}(2\pi\lambda)^2\right) ,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
& e^{i\pi J_0^3} e^{2\pi i\lambda J_0^-} e^{2\pi i z J_0^3} e^{2\pi i\lambda J_0^+} e^{2\pi i z J_0^3} \sim e^{2\pi i\gamma^{(+)}(\lambda,z)J_0^3} , \\
& \gamma^{(+)}(\lambda, z) = \frac{1}{\pi} \arccos\left(\cos\left(\pi\left(2z + \frac{1}{2}\right)\right) + \frac{i}{2}(2\pi\lambda)^2\right) , \\
& \left(e^{i\pi J_0^3} e^{-2\pi i\lambda J_0^-} e^{2\pi i z J_0^3} e^{2\pi i\lambda J_0^+} e^{2\pi i z J_0^3} \sim e^{2\pi i\gamma^{(+)*}(\lambda,z)J_0^3}\right) .
\end{aligned} \tag{3.49}$$

The final result is written as

$$\begin{aligned}
Z_{\text{half}}(\lambda, k) = {} & 2k \int_0^\infty \frac{dt}{t} \frac{V_p}{(8\pi^2 t)^{p/2}} \frac{1}{\eta(it)^8} \frac{1}{2k} \sum_{r\in\mathbf{Z}_{2k}} \times \\
& \times \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^4 (it) \cdot \frac{1}{2}\left\{\theta_3\left(it, i\alpha^{(+)}\left(\lambda, \frac{r}{2k}\right)t\right) e^{-2\pi t\frac{1}{2}\alpha^{(+)}(\lambda,\frac{r}{2k})^2} + \right.\right. \\
& \left. + \theta_3\left(it, i\beta^{(+)}\left(\lambda, \frac{r}{2k}\right)t\right) e^{-2\pi t\frac{1}{2}\beta^{(+)}(\lambda,\frac{r}{2k})^2} \right\} - \\
& - \left(\frac{\theta_4}{\eta}\right)^4 (it) \cdot \frac{1}{2}\left\{\theta_4\left(it, i\alpha^{(+)}\left(\lambda, \frac{r}{2k}\right)t\right) e^{-2\pi t\frac{1}{2}\alpha^{(+)}(\lambda,\frac{r}{2k})^2} - \right.
\end{aligned} \tag{3.50}$$

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$$
\left. - \theta_4\left(it, i\beta^{(+)}\left(\lambda, \frac{r}{2k}\right)t\right) e^{-2\pi t \frac{1}{2}\beta^{(+)}(\lambda, \frac{r}{2k})^2} \right\} -
$$

$$
- \left(\frac{\theta_2}{\eta}\right)^4 (it) \cdot \frac{1}{2} \left\{ \theta_2\left(it, i\gamma^{(+)}\left(\lambda, \frac{r+1/2}{2k}\right)t\right) e^{-2\pi t \frac{1}{2}\gamma^{(+)}(\lambda, \frac{r+1/2}{2k})^2} + \right.
$$

$$
\left. \left. + \theta_2\left(it, i\gamma^{(+)*}\left(\lambda, \frac{r+1/2}{2k}\right)t\right) e^{-2\pi t \frac{1}{2}\gamma^{(+)*}(\lambda, \frac{r+1/2}{2k})^2} \right\} \right] .
$$

Despite the complexity of the angles $\alpha^{(+)}(\lambda, z)$, $\beta^{(+)}(\lambda, z)$ and $\gamma^{(+)}(\lambda, z)$, this is a real function as we can readily confirm it.

4. Effective hagedorn behaviors in the S-brane backgrounds

In this section we argue on the UV behaviors of thermal partition functions we calculated. We first study the full S-brane in a detail, and later discuss the half S-brane case.

4.1 Full S-brane in bosonic string case

We start with focusing on the special points $\lambda = 0$ and $\lambda = 1/2$. The former is nothing but the ordinary time-like Dp-brane. The thermal amplitude (3.18) is identified as the free energy of the open string gas attached at the Dp-brane by the simple relation [41];

$$
F(\beta = 2\pi k) \equiv \frac{1}{\beta} \mathrm{Tr}\left[(-1)^{\mathbf{F}} \ln\left(1 - (-1)^{\mathbf{F}} e^{-\beta p^0}\right)\right] = -\frac{1}{\beta} Z(\lambda = 0, k) . \tag{4.1}
$$

Here $\mathbf{F}$ denotes the space-time fermion number. p^0 is the space-time energy related to the open string hamiltonian $H^{(o)}$ by the on-shell condition as follows;

$$
p^0 \equiv \frac{1}{\sqrt{2}}(p^+ - p^-) = \frac{p^+}{\sqrt{2}} + \frac{H^{(o)}}{2\sqrt{2}p^+} . \tag{4.2}
$$

With the help of this identity the second equality in (4.1) can be directly checked by taking the light-cone gauge. (The light-cone momentum p^+ is converted to the cylinder modulus by the suitable change of variable.) This amplitude (4.1) includes contributions from all the on-shell open string states. On the other hand, in the closed string picture, the boundary state only includes the off-shell states.[5] It is nevertheless quite useful for the investigation of thermodynamical behavior of *physical* open string modes. The Hagedorn temperature [36] is characterized as the temperature over which the *unphysical* closed strings wrapped around the thermal circle become tachyonic [42, 43].

For the latter case, $\lambda = 1/2$, the situation is in a sense reversed. The background reduces to the space-like $D(p-1)$-brane array ("sD-brane" in the terminology of [9]) along the thermal circle (the imaginary time axis) with interval 2π. The cylinder amplitude (3.20)

[5]The boundary state is of course constructed as to be BRST invariant in the corresponding lorentzian theory; $(Q_{\mathrm{BRST}} + \tilde{Q}_{\mathrm{BRST}})|B\rangle = 0$ (or, equivalently, to be conformally invariant in the sense of $(L_n - \tilde{L}_{-n})|B\rangle = 0$). The "off-shell" here means that all the zero-mode momenta appearing in $|B\rangle$ cannot satisfy the mass shell condition for the closed string modes (except for the closed string tachyon). This fact is obvious from the kinematical reason.

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includes on-shell closed string states, but only off-shell states in the open string channel. This feature is likely to be consistent with the fact that we have no physical D-brane on the real time axis at this point.[6] The thermal behavior of physical closed string excitations is captured by the "virtual" open strings wrapped around the thermal circle.

Based on these observations we shall concentrate on (i) near the point $\lambda = 0$ and (ii) near $\lambda = 1/2$. We should argue on the thermal behavior of open string modes in the first case, and the closed string modes in the second case.

(i) Near $\lambda = 0$: effective Hagedorn behavior of open string excitations. As we mentioned above, the simplest way to investigate the thermal behavior of physical open string excitations is to observe the IR behavior of the "dual" closed string picture. By considering the q-expansion of theta functions, we find that the thermal partition function (3.15) has the following structure in the closed string channel (see the second line in (3.14));

$$\begin{aligned} Z(\lambda, k) &= \int_0^\infty ds\, Z^{(c)}(\lambda, k; s)\,, \\ Z^{(c)}(\lambda, k; s) &= \text{const.} \times \frac{1}{s^{(25-p)/2}} \sum_{w=0}^{\infty} Z_w^{(c)}(\lambda, k; s)\,, \end{aligned} \tag{4.3}$$

where w denotes the (absolute value of) "winding number" along the thermal circle originating from the theta functions $\Theta_{*,1}(is, \alpha(\lambda, r/k))$ in (3.14) and $Z_w^{(c)}(\lambda, k; s)$ has the form as

$$Z_w^{(c)}(\lambda, k; s) = e^{2\pi s\left(1-\frac{w^2}{4}\right)} \times \text{power series of } e^{-2\pi s}\,. \tag{4.4}$$

It is obvious that the leading term $Z_0^{(c)}(\lambda, k; s)$ does not depend on the parameters λ, k and behaves as

$$Z_0^{(c)}(\lambda, k; s) = e^{2\pi s} + \mathcal{O}(1)\,, \qquad (s \to \infty) \tag{4.5}$$

which corresponds to the closed string tachyon and is expected to be absent in the superstring case. We are interested in $Z_w^{(c)}(\lambda, k; s)$ ($w \neq 0$). The first non-zero contribution from them captures the desired thermal behavior of open string excitations.

Let us first set $\lambda = 0$. Since $\frac{1}{2}\alpha(0, r/k) = r/k$ holds, the "averaging" of the form $\frac{1}{k}\sum_{r\in\mathbf{Z}} f(r/k)$ kills all the sectors of $w < k$; i.e. $Z_1^{(c)}(0, k; s) = \cdots = Z_{k-1}^{(c)}(0, k; s) = 0$, and the first non-zero term is $Z_k^{(c)}(0, k; s)$. This fact reproduces the thermal compactification $X \sim X + 2\pi k$. To be more precise, we obtain the correct IR behavior at temperature $T = 1/(2\pi k)$;

$$Z^{(c)}(0, k; s) - \frac{1}{s^{(25-p)/2}} Z_0^{(c)}(0, k; s) \sim \frac{1}{s^{(25-p)/2}} e^{2\pi s\left(1-\frac{k^2}{4}\right)}\,, \quad (s \to \infty) \tag{4.6}$$

and no thermal instability appears, if $k > 2$. (Recall that $k = 2$ corresponds to the Hagedorn temperature.)

[6]The interpretation of D-brane instantons located along the imaginary time axis as closed string vertices has been discussed in a detail in the paper [21].

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Turning on the small coupling λ, the situation is drastically changed. In fact, we can evaluate $Z_1^{(c)}(\lambda, k; s)$ as

$$\begin{aligned} Z_1^{(c)}(\lambda, k; s) &= \frac{1}{k} \sum_{r \in \mathbf{Z}_k} \left(e^{2\pi i \alpha(\lambda, r/k)\frac{1}{2}} + e^{-2\pi i \alpha(\lambda, r/k)\frac{1}{2}} \right) e^{2\pi s \cdot \frac{3}{4}} + \mathcal{O}(e^{-2\pi s \cdot \frac{1}{4}}) \\ &= \frac{1}{k} \sum_{r \in \mathbf{Z}_k} 2 \left(1 - 2\sin^2\left(\pi \frac{r}{k}\right) \cos^2(\pi\lambda) \right) e^{2\pi s \cdot \frac{3}{4}} + \mathcal{O}(e^{-2\pi s \cdot \frac{1}{4}}) \\ &= 2\sin^2(\pi\lambda) e^{2\pi s \cdot \frac{3}{4}} + \mathcal{O}\left(e^{-2\pi s \cdot \frac{1}{4}} \right). \end{aligned} \tag{4.7}$$

This leads us to the tachyonic behavior

$$Z^{(c)}(\lambda, k; s) - \frac{1}{s^{(25-p)/2}} Z_0^{(c)}(\lambda, k; s) \sim 2\sin^2(\pi\lambda) \frac{1}{s^{(25-p)/2}} e^{2\pi s \cdot \frac{3}{4}}, \qquad (s \to \infty), \tag{4.8}$$

characteristic for the Hagedorn divergence at the very high temperature $T = 2T_H$. We note that this behavior (4.8) is universal *irrespective of the temperature* $T = 1/(2\pi k)$. Namely, we always face this divergence no matter how low temperature (including the zero-temperature $k = \infty$) is chosen, as long as the coupling λ is non-zero. Such effective thermalization as if we were at a very high temperature caused by the S-brane has been discussed in [5, 9] based on the minisuperspace approximation, and it has been explained as the reflection of large rate of open string pair production observed by the Unruh detector. Our result is likely to be consistent with this observation at the qualitative level. However, we should note that the production rate calculated in [5, 9] is characterized by T_H rather than $2T_H$.

(ii) Near $\lambda = 1/2$: effective Hagedorn behavior of closed string excitations. As mentioned above, closed string physical states contribute to the thermal partition function (3.20) at the point $\lambda = 1/2$. The thermal instability caused by closed string excitations is formally examined in the dual open string channel despite the absence of physical open string states. We define

$$Z(\lambda, k) = \int_0^\infty dt\, Z^{(o)}(\lambda, k; t). \tag{4.9}$$

Recalling (3.15), we find that $Z^{(o)}(\lambda, k; t)$ has the following structure

$$Z^{(o)}(\lambda, k; t) = \text{const.} \times \frac{1}{t^{1+p/2}} \sum_{w=0}^{\infty} Z_w^{(o)}(\lambda, k; t), \tag{4.10}$$

where $Z_w^{(o)}(\lambda, k; t)$ has the form as

$$Z_w^{(o)}(\lambda, k; t) = \sum_{r \in \mathbf{Z}_k} \left[e^{2\pi t \left\{ 1 - \left(w + \frac{1}{2}\alpha(\lambda, r/k) \right)^2 \right\}} + e^{2\pi t \left\{ 1 - \left(w - \frac{1}{2}\alpha(\lambda, r/k) \right)^2 \right\}} \right] \text{power series of } e^{-2\pi t}. \tag{4.11}$$

The leading term $Z_0^{(o)}(\lambda, k; t)$ is again tachyonic irrespective of λ, k, and not relevant for our analysis. The next leading term $Z_1^{(o)}(\lambda, k; t)$ is important for the thermal property. We first set $\lambda = 1/2$. Recalling (3.20), we find $Z_1^{(o)}(1/2, k; t)$ behaves as

$$Z_1^{(o)}(1/2, k; t) \sim 1, \qquad (t \to \infty), \tag{4.12}$$

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since $\alpha(1/2, z) = 0$ holds. It means that we have no thermal instability in the closed string channel at least for $p \geq 1$. This fact is quite natural because $\lambda = 1/2$ means that we sit at a minimum of the tachyon potential from the beginning and no particle production seems to be caused.

Now let us consider a small perturbation from the $\lambda = 1/2$ point. We set $\lambda = \frac{1}{2} - \Delta\lambda$, $0 < \Delta\lambda \ll 1$. Based on the simple evaluation

$$\alpha\left(\frac{1}{2} - \Delta\lambda, z\right) = 2\sin(\pi z)\Delta\lambda + \mathcal{O}((\Delta\lambda)^3)\,, \tag{4.13}$$

we can find the tachyonic behavior

$$Z_1^{(o)}\left(\frac{1}{2} - \Delta\lambda, k; t\right) \sim e^{2\pi t \Delta\lambda'(2-\Delta\lambda')}\,, \qquad (t \to \infty)\,, \tag{4.14}$$

where we set $\Delta\lambda' \equiv \Delta\lambda \sin\left(\frac{\pi}{k}\left[\frac{k}{2}\right]\right)$ ([ ] is the Gauss symbol). The IR divergence included in $Z_1^{(o)}(1/2 - \Delta\lambda, k; t)$ is interpreted in the physical closed string picture as the large number of excitations of *all* the closed string massive modes as if we were above the Hagedorn temperature. We again note that this feature does not depend on k, and hence we always face such instability even in the zero temperature case. It has been shown in [8] that we have the large emission rate of closed string massive modes in the brane decay process, which gives rise to a Hagedorn like divergence. Our observation seems to be consistent with this fact. One should, however, keep it in mind that we have calculated the euclidean cylinder amplitudes, which has different physical meaning compared with that of the lorentzian signature treated in [8] (see also [27]) in spite of the apparent similarity of structure as the BCFT amplitudes.

4.2 Full S-brane in superstring case

The superstring case is similarly analysed. We will observe the essentially same thermodynamical behavior, but obtain a little modification.

(i) Near $\lambda = 0$: effective Hagedorn behavior of the open string excitations We again start with exploring the thermal behavior in the open string channel around $\lambda = 0$ point. We define

$$\begin{aligned} Z(\lambda, k) &= \int_0^\infty dt\, Z^{(c)}(\lambda, k; s)\,, \\ Z^{(c)}(\lambda, k; s) &= \text{const.} \times \frac{1}{s^{(9-p)/2}} \sum_{w=0}^\infty Z_w^{(c)}(\lambda, k; s)\,, \end{aligned} \tag{4.15}$$

analogously to (4.3). The winding w is defined associated to the level 2 theta functions $\theta_*(is, *)$ (see (3.39), (3.40) and (3.43)), and $Z_w^{(c)}(\lambda, k; s)$ has the form

$$Z_w^{(c)}(\lambda, k; s) = e^{2\pi s \cdot \frac{1}{2}(1-w^2)} \times \text{power series of } e^{-2\pi s \cdot \frac{1}{2}}\,. \tag{4.16}$$

The $w = 0$ sector behaves as

$$Z_0^{(c)}(\lambda, k; s) \sim 1\,, \qquad (s \to \infty)\,, \tag{4.17}$$

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since the closed string tachyon mode is eliminated by the GSO projection. If $\lambda = 0$, since $\alpha(0, r/(2k)) = r/k$, $\beta(0, r/(2k)) = r/k$, $\gamma(0, (r+1/2)/(2k)) = (r+1/2)/k + 1/2$ hold, the averaging $1/2k \sum_{r\in \mathbf{Z}_{2k}} f(r/k)$ kills all the sectors of $w = 1, \ldots, k-1$. The first non-zero term $Z_k^{(c)}(0, k; t)$ behaves as

$$
\begin{aligned}
Z_k^{(c)}(0,k;s) &\sim e^{-2\pi s\cdot\frac{1}{2}k^2} \frac{1}{\eta(is)^8} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 (is) + \left(\frac{\theta_4}{\eta}\right)^4 (is) \right\} \\
&\sim e^{2\pi s\cdot \frac{1}{2}(1-k^2)}, \qquad (s \to \infty)\,.
\end{aligned} \tag{4.18}
$$

We so obtain

$$
Z^{(c)}(0,k;s) - \frac{1}{s^{(9-p)/2}} Z_0^{(c)}(0,k;s) \sim \frac{1}{s^{(9-p)/2}} e^{2\pi s\cdot\frac{1}{2}(1-k^2)}\,, \qquad (s\to\infty)\,, \tag{4.19}
$$

reproducing the correct thermal behavior at $T = 1/(2\pi\sqrt{2}k)$. Note that $k=1$ corresponds to the Hagedorn temperature for superstring (while $T_{k=1} = 2T_H$ in bosonic string).

Turning on the coupling λ, we obtain as in (4.7)

$$
\begin{aligned}
Z_1^{(c)}(\lambda,k;s) &= \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}} \left[\frac{1}{2}\left\{ e^{2\pi i\alpha(\lambda,r/(2k))} + e^{-2\pi i\alpha(\lambda,r/(2k))} + e^{2\pi i\beta(\lambda,r/(2k))} + e^{-2\pi i\beta(\lambda,r/(2k))} \right\} + \right. \\
&\qquad \left. + e^{2\pi i\gamma(\lambda,(r+1/2)/(2k))} + e^{-2\pi i\gamma(\lambda,(r+1/2)/(2k))} \right] + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{2}}\right) \\
&= \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}} \left[-8\cos^2(\pi\lambda) + 8\left\{ \sin^4\left(\pi\frac{r}{2k}\right) + \cos^4\left(\pi\frac{r}{2k}\right)\right\}\cos^4(\pi\lambda) + \right. \\
&\qquad \left. + 4\sin^2\left(2\pi\frac{r+1/2}{2k}\right)\cos^4(\pi\lambda)\right] + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{2}}\right) \\
&= -4\sin^2(2\pi\lambda) + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{2}}\right).
\end{aligned} \tag{4.20}
$$

This leads us to the massless IR behavior

$$
Z^{(c)}(\lambda,k;s) - \frac{1}{s^{(9-p)/2}} Z_0^{(c)}(\lambda,k;s) \sim -4\sin^2(2\pi\lambda)\frac{1}{s^{(9-p)/2}}\,, \qquad (s\to\infty)\,, \tag{4.21}
$$

which means that the open string excitations behave as if we were at the Hagedorn temperature not depending on k. In this way we have shown the effective thermalization with non-zero λ as in the bosonic string case. We have, however, a "moderate" thermal behavior compared with the bosonic string. Namely, the behavior (4.21) is massless rather than tachyonic, and hence the t-integral has no IR divergence as long as $p \leq 6$. It is of course originating from the (partial) SUSY cancellation left even under the thermal boundary condition of world-sheet fermions.

(ii) Near $\lambda = 1/2$: effective Hagedorn behavior of the closed string excitations
We can again analyse likewise as in the bosonic string. We write

$$
Z(\lambda,k) = \int_0^\infty dt\, Z^{(o)}(\lambda,k;t)\,. \tag{4.22}
$$

At $\lambda = 1/2$, (3.46) leads to the massless behavior

$$Z^{(o)}(1/2, k; t) \sim \frac{1}{t^{p/2+1}}, \qquad (t \to \infty). \tag{4.23}$$

As is expected, we have no tachyonic divergence, and the closed string channel has no thermal instability if $p \geq 1$.

Let us now consider the small perturbation $\lambda = 1/2 - \Delta\lambda$ ($0 < \Delta\lambda \ll 1$). The most important parts included in the thermal partition function (3.44) are the theta functions such as $\theta_*(is, i\alpha(\lambda,*)t)e^{-2\pi t\cdot\frac{1}{2}\alpha(\lambda,z)^2}$, which yields the contribution

$$\sim \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}}\left[e^{-2\pi t\cdot\frac{1}{2}\alpha(\lambda,r/(2k))^2} + \sum_{w=1}^{\infty}\left\{e^{-2\pi t\cdot\frac{1}{2}(w+\alpha(\lambda,r/(2k)))^2} + e^{-2\pi t\cdot\frac{1}{2}(w-\alpha(\lambda,r/(2k)))^2}\right\}\right], \tag{4.24}$$

and the similar terms including $\beta(\lambda,*)$ and $\gamma(\lambda,*)$. The $w=1$ term is relevant for the desired thermal behavior and actually gives the leading term. (The thermal boundary condition for fermions is essential for this fact.) Based on the evaluations

$$\begin{aligned}
\alpha\left(\frac{1}{2}-\Delta\lambda, z\right) &= 2\sin(\pi z)\Delta\lambda + \mathcal{O}\left((\Delta\lambda)^3\right), \\
\beta\left(\frac{1}{2}-\Delta\lambda, z\right) &= 1 - 2\cos(\pi z)\Delta\lambda + \mathcal{O}\left((\Delta\lambda)^3\right), \\
\gamma\left(\frac{1}{2}-\Delta\lambda, z\right) &= \frac{1}{2} + \pi\sin(2\pi z)(\Delta\lambda)^2 + \mathcal{O}\left((\Delta\lambda)^4\right),
\end{aligned} \tag{4.25}$$

we find the tachyonic behavior

$$Z^{(o)}\left(\frac{1}{2}-\Delta\lambda, k; t\right) \sim \frac{1}{t^{1+p/2}}e^{4\pi t\Delta\lambda(1-\Delta\lambda)}, \qquad (t \to \infty). \tag{4.26}$$

In this way we have again found the effective thermal instability in the closed string channel.

4.3 Half S-branes

As we already mentioned, in the half S-brane case it is natural to rewrite the coupling λ as $\lambda \equiv \lambda_0 e^{x^0}$ and identify x^0 as the real time. We assume that λ_0 is fixed to be a positive number of $\mathcal{O}(1)$, and discuss the thermal behaviors in the far past ($x^0 \sim -\infty$) and in the far future ($x^0 \sim +\infty$).

In the far past we should discuss the thermal behavior of on-shell open string states and the analysis is quite similar to that for the full S-brane near the $\lambda = 0$ point. In the bosonic string case, (4.7) is replaced with

$$\begin{aligned}
Z_1^{(c)}(\lambda_0 e^{x^0}, k; s) &= \frac{1}{k}\sum_{r\in\mathbf{Z}_k}\left(e^{2\pi i\alpha^{(+)}(\lambda_0 e^{x^0}, r/k)\frac{1}{2}} + e^{-2\pi i\alpha^{(+)}(\lambda_0 e^{x^0}, r/k)\frac{1}{2}}\right)e^{2\pi s\cdot\frac{3}{4}} + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{4}}\right) \\
&= \frac{1}{k}\sum_{r\in\mathbf{Z}_k}\left[(2\pi\lambda_0 e^{x^0})^2 + 2\cos\left(2\pi\frac{r}{k}\right)\right]e^{2\pi s\cdot\frac{3}{4}} + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{4}}\right) \\
&= \left(2\pi\lambda_0 e^{x^0}\right)^2 e^{2\pi s\cdot\frac{3}{4}} + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{4}}\right),
\end{aligned} \tag{4.27}$$

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which yields the same tachyonic behavior as in (4.7). On the other hand, in the superstring case, we obtain in the similar manner as (4.20)

$$\begin{aligned}
Z_1^{(c)}(\lambda_0 e^{x^0}, k; s) &= \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}}\frac{1}{2}\Big[e^{2\pi i\alpha^{(+)}(\lambda_0 e^{x^0}, r/(2k))} + e^{-2\pi i\alpha^{(+)}(\lambda_0 e^{x^0}, r/(2k))} + \\
&\qquad + e^{2\pi i\beta^{(+)}(\lambda_0 e^{x^0}, r/(2k))} + e^{-2\pi i\beta^{(+)}(\lambda_0 e^{x^0}, r/(2k))} \\
&\qquad + e^{2\pi i\gamma^{(+)}(\lambda_0 e^{x^0}, (r+1/2)/(2k))} + e^{-2\pi i\gamma^{(+)}(\lambda_0 e^{x^0}, (r+1/2)/(2k))} \\
&\qquad + e^{2\pi i\gamma^{(+)*}(\lambda_0 e^{x^0}, (r+1/2)/(2k))} + e^{-2\pi i\gamma^{(+)*}(\lambda_0 e^{x^0}, (r+1/2)/(2k))}\Big] + \\
&\quad +\mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{2}}\right) \\
&= \frac{1}{2k}\sum_{r\in\mathbf{Z}_{2k}}\left[4\cos^2\left(2\pi\frac{r}{2k}\right) + 4\sin^2\left(2\pi\frac{r+\frac{1}{2}}{2k}\right) - 4\right] \\
&= 0 + \mathcal{O}\left(e^{-2\pi s\cdot\frac{1}{2}}\right).
\end{aligned} \tag{4.28}$$

We have thus found a massive behavior not depending on x^0. Interestingly, we have the cancellation of the massless term in contrast to the full brane case (4.20).

In the far future, the situation is drastically changed. We should now examine the thermal behavior of closed string physical excitations. The twist angles $\alpha^{(+)}(\lambda_0 e^{x^0}, z)$, $\beta^{(+)}(\lambda_0 e^{x^0}, z)$, and $\gamma^{(+)}(\lambda_0 e^{x^0}, z)$ gain very large imaginary parts, explicitly evaluated as

$$\begin{aligned}
\alpha^{(+)}(\lambda_0 e^{x^0}, z) &\sim -i\frac{2}{\pi}x^0\,, \\
\beta^{(+)}(\lambda_0 e^{x^0}, z) &\sim 1 - i\frac{2}{\pi}x^0\,, \\
\gamma^{(+)}(\lambda_0 e^{x^0}, z) &\sim \frac{1}{2} - i\frac{2}{\pi}x^0\,.
\end{aligned} \tag{4.29}$$

We so obtain

$$Z_{\mathrm{half}}(\lambda_0 e^{x^0}, k) \approx k\int_0^\infty \frac{dt}{t}\frac{V_p}{(8\pi^2 t)^{p/2}}\frac{1}{\eta(it)^{24}}\sum_{n\in\mathbf{Z}} e^{-\frac{t}{2\pi}\left\{2\pi\left(n-\frac{i}{\pi}x^0\right)\right\}^2}, \tag{4.30}$$

for bosonic string, and

$$Z_{\mathrm{half}}(\lambda_0 e^{x^0}, k) \approx 2k\int_0^\infty \frac{dt}{t}\frac{V_p}{(8\pi^2 t)^{p/2}}\frac{1}{\eta(it)^{8}}\left(\frac{\theta_4}{\eta}\right)^4(it)\sum_{n\in\mathbf{Z}} e^{-\frac{t}{2\pi}\left\{2\sqrt{2}\pi\left(n+\frac{1}{2}-\frac{i}{\pi}x^0\right)\right\}^2}, \tag{4.31}$$

for superstring. They have quite reminiscent forms of the $\lambda = 1/2$ amplitudes in the full S-branes (3.20) and (3.46), which seems consistent with the expectation that we only have physical closed string modes after the unstable brane collapsed. However, we now have a very strong divergence growing exponentially with respect to $(x^0)^2$.[7] In a naive sense this

[7] Also in the full S-brane case we can introduce the time dependence by replacing the boundary interaction $\lambda\int_{\partial\Sigma} d\tau\,\cos X$ with $\frac{\lambda_0}{2}\int_{\partial\Sigma} d\tau\,\left(e^{x^0}e^{iX} + e^{-x^0}e^{-iX}\right) \sim \frac{\lambda_0}{2}\cdot 2\pi i\left(e^{x^0}J_0^+ + e^{-x^0}J_0^-\right)$. For the far future and past $x^0 \sim \pm\infty$, the twist angles likewise gain large imaginary parts, giving rise to the same type divergence.

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strong divergence may be interpreted as the infinite contribution from the massive closed string excitations emitted from the decaying brane. We will again discuss this point in the last section.

4.4 Comments on space-like linear dilaton backgrounds

Quite recently the S-branes in the general linear dilaton backgrounds have been studied in [27] (see also [7, 22, 23, 25, 26]). Among other things, the UV divergences in the lorentzian cylinder amplitudes have been shown to be removed by considering the space-like linear dilaton characteristic for the subcritical string theories. Let us briefly discuss whether or not the same mechanism to remove the divergence works in our cases of thermal amplitudes.

We only focus on the bosonic string case and the superstring case is similarly worked out (except for a little modification due to the GSO projection). Consider the general conformal system of the form;

$$X \otimes \mathbf{R}_\phi \otimes \mathcal{M}\,, \tag{4.32}$$

where X is the euclidean time coordinate as before and $\mathbf{R}_\phi$ expresses the CFT with the linear dilaton $\Phi(\phi) = Q\phi$ ($Q \in \mathbf{R}$). $\mathcal{M}$ is assumed to be an arbitrary unitary CFT. The criticality condition now becomes

$$1 + \left(1 + 6Q^2\right) + c_{\mathcal{M}} = 26\,. \tag{4.33}$$

We assume for simplicity that the boundary interaction is introduced only along the euclidean time direction X.[8] Namely, we take the world-sheet action (3.2) for the X-sector. The criticality condition leads to the upper bound $Q^2 \leq 4$, since $c_{\mathcal{M}}$ must be non-negative.

Now, let us argue on how the thermal behavior is modified by the influence of linear dilaton. The idea is very simple. In general the cylinder amplitude in an arbitrary BCFT behaves as

$$Z(is) \sim e^{-2\pi s \cdot \frac{c_{\text{eff}}}{24}}\,, \qquad (s \to \infty)\,, \tag{4.34}$$

where s is the closed string or open string modulus (up to some power factor of s in which we are not interested here). The "effective central charge" c_{eff} is defined in [44] (see also [45]);

$$c_{\text{eff}} \equiv c - 24h_0\,, \tag{4.35}$$

where h_0 is the minimal value of conformal weight in the spectrum of normalizable states. For any unitary CFT we have $c_{\text{eff}} = c$, since $h_0 = 0$. However, in the linear dilaton case we have the mass gap $h_0 = Q^2/4 > 0$ and hence obtain

$$c_{\text{eff}} = (1 + 6Q^2) - 24 \times \frac{Q^2}{4} = 1\,. \tag{4.36}$$

[8]In fact, if $Q^2 < 4$, we cannot introduce the suitable Liouville potential for ϕ satisfying the reality condition as is familiar in the context of two dimensional gravity. In the exceptional case $Q^2 = 4$ (in other words, with no $\mathcal{M}$ sector) we can consider the (boundary) Liouville potential both along the time direction and the ϕ-direction. This case has been intensively studied in the recent papers [22, 23, 25].

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In our previous analysis it amounts to replacing the tachyonic factor $e^{2\pi s}$ with an weaker one $e^{2\pi s(1-\frac{Q^2}{4})} \equiv e^{2\pi s\frac{c_{\mathcal{M}}}{24}}$. For example, (4.7) is now modified as

$$Z_1^{(c)}(\lambda, k; s) = 2\sin^2(\pi\lambda)e^{2\pi s\cdot(\frac{3}{4}-\frac{Q^2}{4})} + \mathcal{O}\left(e^{-2\pi s\cdot(\frac{1}{4}+\frac{Q^2}{4})}\right). \tag{4.37}$$

Therefore, the effective Hagedorn divergence is removed for sufficiently large Q (for sufficiently small $c_{\mathcal{M}}$, in other words). The same mechanism to reduce the divergence works in the other cases. However, since Q^2 is at most a number of $\mathcal{O}(1)$, we can never remove the strong divergences appearing in (4.30) and (4.31) (the far future amplitude in the half S-brane background describing the brane decay).

5. Discussions

In this paper we have calculated the thermal partition functions for the S-brane backgrounds. We especially examined the thermal property of the full S-brane case in a detail, and showed that we always have the Hagedorn like divergence no matter how low temperature is taken, if the boundary coupling λ is non-zero. Parts of such divergences can be removed by considering the space-like linear dilaton backgrounds, but cannot for the far future amplitude in the half S-brane background corresponding to the brane decay.

The appearance of UV divergences in the thermal partition functions is likely to be a signal of the infinite contribution to the free energy from the massive closed string excitations emitted from the decaying brane. However, to be more rigorous it may imply the failure of perturbative calculation in string theory. In fact, despite the true marginality and exact solubility, the appearance of divergence in the modulus integral compels us to introduce a cut-off parameter, which breaks the conformal invariance. In string theory the UV divergence in the closed (open) string channel should be interpreted as the wrong choice of vacuum in the dual open (closed) string channel [46, 47, 48]. Such "back-reaction" problem was partly discussed in [8], and it was suggested that the $\lambda = 1/2$ point in the full S-brane may be the solution correctly incorporating the back-reaction. It might be amazing and suggestive that our divergent amplitudes of half S-brane in the far future (4.30) and (4.31) have the reminiscent forms of the $\lambda = 1/2$ amplitudes (3.20) and (3.46). We also point out that the $\lambda = 1/2$ superstring amplitude (3.46) is actually finite (for generic choice of p), suggesting the correctly chosen vacuum. In any case, the appearance of Hagedorn like divergence seems to suggest that the stringy correction is not small and the back-reaction is not negligible *even under the weak coupling limit* $g_s \to 0$. (In fact, we observed the divergence in the thermal partition function as free string theory.) To study the back-reaction seriously is surely the most important and challenging task and we deserve it for future study.

It is an also interesting subject to perform the similar thermal analysis in the general linear dilaton backgrounds. Especially, it is a non-trivial problem to ask whether our calculation of thermal amplitudes can be extended to the S-brane backgrounds with the non-vanishing time component of linear dilaton. Presumably, the "boundary minimal models" would play important roles, and the quantum SU(2) algebra might provide useful tools of calculation.

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A. Some notations

We here summarize the convention of theta functions. We set $q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi iz}$.

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \\
&\equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m)\,, \\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m)\,, \\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}), \\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2})\,. \\
\Theta_{m,k}(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}\,,
\end{aligned} \tag{A.1}$$

We often use the abbreviations; $\theta_i \equiv \theta_i(\tau,0)$ ($\theta_1 \equiv 0$), $\Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau,0)$. We also use the standard convention of η-function;

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{A.2}$$

The affine character of $SU(2)_k$ with spin $\ell/2$ ($0 \le \ell \le k$) is given by the formula

$$\chi_\ell^{(k)}(\tau,z) = \frac{\Theta_{\ell+1,k+2}(\tau,z)-\Theta_{-\ell-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)}\,. \tag{A.3}$$

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